$\mathbb{R}$eal Number Proving in PVS

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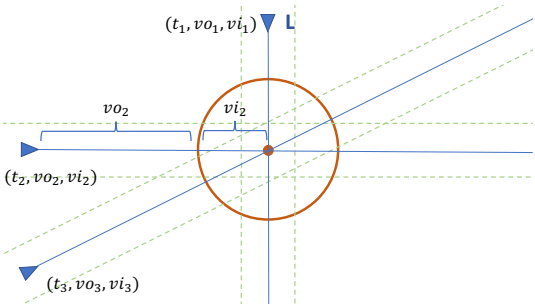
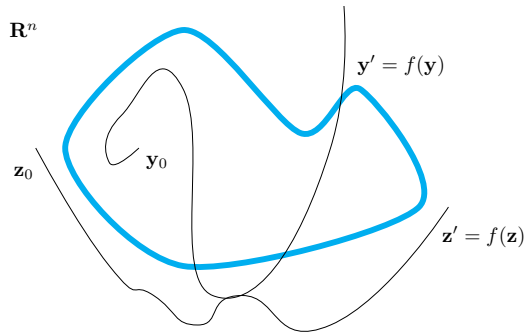
Outline

Why Real Number Proving in PVS ?





Why Real Number Proving in PVS ?

- Real numbers appear in *real* applications, i.e., cyber-physical systems.
- Conceptually, it is easier to reason on a continuous framework than on a discrete one.
- Availability of many classical results in calculus, trigonometry, and continuous mathematics.

Computer Algebra Systems (CAS) and Theorem Provers

- Mathematica, Maple, Matlab, etc. provide very powerful symbolic and numerical engines.
- These systems **do not** claim to be *logically sound*. Singularities and exceptions are well-known problems of CAS.
- CAS provide programming languages (as opposed to *specification languages*.)
- Real analysis is not a traditional strength of theorem provers.
  - CAS can be used to perform mechanical simplifications and find potential solutions.
  - A theorem prover can be used to verify the correctness of a particular solution.

Real Numbers in PVS

- Reals are defined as an uninterpreted subtype of `number_field` in the prelude library:
  ```
  real: NONEMPTY_TYPE FROM number_field
  ```
- All numeric constants are `real`:
  - naturals: `0,1,...`
  - integers: `...,-1,0,1,...`
  - rationals: `...,-1/10,...,3/2,...`

- Decimal notation is supported: The decimal number `3.141516` is syntactic sugar for the rational number `31416/10000`.

PVS's `real` numbers are $\mathbb{R}$eal

- All the standard properties: unbounded, connected, infinite, $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, ...
- $\mathbb{R}$eal arithmetic: `1/3 + 1/3 + 1/3 = 1`.
- The type `real` is unbounded:

```
googol:  real = 10 ^ 100

googolplex:  real = 10 ^ googol

googol_prop :  LEMMA
    googolplex > googol * googol
```

- ... but *machine physical limitations do apply*, e.g., don't try to prove `googol_prop` with `(grind)`.

Rational Arithmetic is Built-in

```
   |-------
{1} -(0.78 * 1.05504 * (0.92 - 0.78) * s) -
      0.78 * 1.08016 * (0.9 - 0.78) * s
       - 1.256 * (0.9 -0.78) * s * u
       - 0.92944 * (0.92 - 0.78) * s * u
      + 1.05504 * (0.92 - 0.78) * s * u
       + 1.08016 * (0.9 - 0.78) * s * u >= 0

>>
```

Rational Arithmetic is Built-in

```
  |-------
{1} -(0.78 * 1.05504 * (0.92 - 0.78) * s) -
      0.78 * 1.08016 * (0.9 - 0.78) * s
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       - 0.92944 * (0.92 - 0.78) * s * u
      + 1.05504 * (0.92 - 0.78) * s * u
       + 1.08016 * (0.9 - 0.78) * s * u >= 0

>> (assert)
  |-------
{1} 55107/3906250 * s - 1099 / 312500 * (s * u) >= 0
```

Subtypes of `real`

```
nzreal :  TYPE+ = {r:real| r /= 0} % Nonzero reals
nnreal :  TYPE+ = {r:real| r >= 0} % Nonnegative reals
npreal :  TYPE+ = {r:real| r <= 0} % Nonpositive reals
negreal :  TYPE+ = {r:real| r < 0} % Negative reals
posreal :  TYPE+ = {r:real| r > 0} % Positive reals

rat :  TYPE+ FROM real
int :  TYPE+ FROM rat
nat :  TYPE+ FROM int
```

The uninterpreted type `number` *is* `real`*'s largest supertype predefined in PVS: allows for definition of complex numbers, hyper-reals,* $\mathbb{R}^{\infty}$*, . . . , as supertypes of reals*

Real Numbers Properties

Real numbers in PVS are axiomatically defined in the prelude:

- `real_axioms: THEORY`
  Commutativity, associativity, identity, etc. These properties are known to the decision procedures, so they rarely need to be explicitly introduced in a proof.

- `real_props: THEORY`
  Order and cancellation laws. These lemmas are not used automatically by the standard decision procedures.

Theory `real_props`

```
real_props: THEORY
BEGIN
both_sides_plus_le1: LEMMA x + z <= y + z IFF x <= y
both_sides_plus_le2: LEMMA z + x <= z + y IFF x <= y
both_sides_minus_le1: LEMMA x - z <= y - z IFF x <= y
both_sides_minus_le2: LEMMA z - x <= z - y IFF y <= x
both_sides_div_pos_le1: LEMMA x/pz <= y/pz IFF x <= y
both_sides_div_neg_le1: LEMMA x/nz <= y/nz IFF y <= x
...
abs_mult: LEMMA abs(x * y) = abs(x) * abs(y)
abs_div: LEMMA abs(x / n0y) = abs(x) / abs(n0y)
abs_abs: LEMMA abs(abs(x)) = abs(x)
abs_square: LEMMA abs(x * x) = x * x
abs_limits: LEMMA -(abs(x) + abs(y)) <= x + y AND
                  x + y <= abs(x) + abs(y)
END real_props
```

Predefined Operations

```
+, -, *:  [real, real -> real]
/:  [real, nzreal -> real]
-:  [real -> real]

sgn(x:real) :  int = IF x >= 0 THEN 1 ELSE -1 ENDIF
abs(x:real) :  {nny:  nnreal | nny >= x} = ...
max(x,y:real):  {z:  real | z >= x AND z >= y} = ...
min(x,y:real):  {z:  real | z <= x AND z <= y} = ...
^(x:real, i:{i:int | x /= 0 OR i >= 0}):  real = ...
```

. . . and what about $\sqrt{}, \int, \log, \exp, \sin, \cos, \tan, \pi, \lim, \ldots$ *?*

NASA PVS Libraries

http://github.com/nasa/pvslib

- `reals`: Square, square root, quadratic formula, polynomials.
- `analysis`: Real analysis, limits, continuity, derivatives, integrals.
- `vectors` and `vect_analysis`: Vector calculus and analysis.
- `series`: Power series, Taylor's theorem.
- `trig`: Trigonometric functions.
- `lnexp_fnd`: Logarithm, exponential, and hyperbolic functions.

Beyond Real Numbers

- `complex` and `complex_alt`: Complex numbers.
- `float`: Floating point numbers.
- `interval_arith`: Interval arithmetic.
- `affine_arith`: Affine arithmetic.
- `exact_real_arith`: Exact real arithmetic.
- ...

Basic Real Number Proving

PVS offers some proof commands for simple algebraic manipulations:

```
one_fourth :

  |-------
{1} x - x * x <= 1

>> (both-sides "-" "1/4")
one_fourth :

  |-------
{1} x - x * x - 1 / 4 <= 1 - 1 / 4
```

Note: Use `both-sides` only to add/subtract expressions.

Basic Real Number Proving

PVS offers some proof commands for simple algebraic manipulations:

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one_fourth :

  |-------
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{1} x - x * x - 1 / 4 <= 1 - 1 / 4
```

Note: Use `both-sides` only to add/subtract expressions.

Use `case` to Prove What You Need

```
  |-------
{1} x - x * x - 1 / 4 <= 1 - 1 / 4

>> (case "x - x * x - 1 / 4 <= 0")
this yields 2 subgoals:
one_fourth.1 :
{-1} x - x * x - 1 / 4 <= 0
  |-------
[1] x - x * x - 1 / 4 <= 1 - 1 / 4

>> (assert)
This completes the proof of one_fourth.1.
```

Use `case` to Prove What You Need

```
  |-------
{1} x - x * x - 1 / 4 <= 1 - 1 / 4

>> (case "x - x * x - 1 / 4 <= 0")
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Use `case` to Prove What You Need

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  |-------
{1} x - x * x - 1 / 4 <= 1 - 1 / 4

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this yields 2 subgoals:
one_fourth.1 :
{-1} x - x * x - 1 / 4 <= 0
  |-------
[1] x - x * x - 1 / 4 <= 1 - 1 / 4

>> (assert)
This completes the proof of one_fourth.1.
```

Use `hide` to Focus on Relevant Formulas

```
one_fourth.2 :
  |-------
{1} x - x * x - 1 / 4 <= 0
[2] x - x * x - 1 / 4 <= 1 - 1 / 4

>> (hide 2)
one_fourth.2 :
  |-------
[1] x - x * x - 1 / 4 <= 0
```

Use `hide` to Focus on Relevant Formulas

```
one_fourth.2 :
  |-------
{1} x - x * x - 1 / 4 <= 0
[2] x - x * x - 1 / 4 <= 1 - 1 / 4

>> (hide 2)
one_fourth.2 :
  |-------
[1] x - x * x - 1 / 4 <= 0
```

Arrange Expressions With `case-replace`

```
one_fourth.2 :
  |-------
[1] x - x * x - 1 / 4 <= 0

>> (case-replace
      "x - x * x - 1 / 4 = -(x-1/2)*(x-1/2)"
      :hide? t)
this yields 2 subgoals:
one_fourth.2.1 :
  |-------
{1} -(x - 1 / 2) * (x - 1 / 2) <= 0
```

Arrange Expressions With `case-replace`

```
one_fourth.2 :
  |-------
[1] x - x * x - 1 / 4 <= 0

>> (case-replace
      "x - x * x - 1 / 4 = -(x-1/2)*(x-1/2)"
      :hide? t)
this yields 2 subgoals:
one_fourth.2.1 :
  |-------
{1} -(x - 1 / 2) * (x - 1 / 2) <= 0
```

Introduce New Names With `name-replace`

```
one_fourth.2.1 :
  |-------
{1} -(x - 1 / 2) * (x - 1 / 2) <= 0

>> (name-replace "X" "(x-1/2)")
one_fourth.2.1 :
  |-------
{1} -X * X <= 0

>> (assert)
This completes the proof of one_fourth.2.1.
```

Introduce New Names With `name-replace`

```
one_fourth.2.1 :
  |-------
{1} -(x - 1 / 2) * (x - 1 / 2) <= 0

>> (name-replace "X" "(x-1/2)")
one_fourth.2.1 :
  |-------
{1} -X * X <= 0

>> (assert)
This completes the proof of one_fourth.2.1.
```

Introduce New Names With `name-replace`

```
one_fourth.2.1 :
  |-------
{1} -(x - 1 / 2) * (x - 1 / 2) <= 0

>> (name-replace "X" "(x-1/2)")
one_fourth.2.1 :
  |-------
{1} -X * X <= 0

>> (assert)
This completes the proof of one_fourth.2.1.
```

Don't Reinvent the Wheel

Look into the NASALib first!

Theory `reals@quadratic`:

```
quadratic_le_0 :  LEMMA
a*sq(x) + b*x + c <= 0 IFF
((discr(a,b,c) >= 0 AND
   ((a > 0 AND x2(a,b,c) <= x AND x <= x1(a,b,c)) OR
   (a < 0 AND (x <= x1(a,b,c) OR x2(a,b,c) <= x)))) OR
   (discr(a,b,c) < 0 AND c <= 0))
```

A Simpler Proof

```
   |-------
{1} x * (1 - x) <= 1

>> (lemma "quadratic_le_0"
    ("a" "-1" "b" "1" "c" "-1" "x" "x"))
   (grind)
Trying repeated skolemization, instantiation, and
if-lifting,
Q.E.D.
```

An Even Simpler Proof

```
   |-------
{1} x * (1 - x) <= 1

>> (sturm)
Q.E.D.
```

Manip

- Manip is a PVS package for algebraic manipulations of real-valued expressions.
- `http://shemesh.larc.nasa.gov/people/bld/manip.html`.
- The package consists of:
  - Strategies.
  - Extended notations for formulas and expressions.
  - Support functions for strategy developers.
- Fully integrated into PVS

Manip Strategies: Basic Manipulations

Strategy	Description
`(swap-rel` *fnums*`)`	Swap sides and reverse relations
`(swap!` *exprloc*`)`	$x \circ y \Rightarrow y \circ x$
`(group!` *exprloc* `1|r)`	$(x \circ y) \circ z \Rightarrow x \circ (y \circ z)$
`(flip-ineq` *fnums*`)`	Negate and move inequalities
`(split-ineq` *fnum*`)`	Split $\leq$ ($\geq$) into $<$ ($>$) and $=$

Extended Formula Notation

- Standard
  - `*`: All formulas.
  - `-`: All formulas in the antecedent.
  - `+`: All formulas in the consequent.

- Extended (Manip strategies only)
  - `(^ n1...nk)`: All formulas but $n_1, \ldots, n_k$
  - `(-^ n1...nk)`: All antecedent formulas but $n_1, \ldots, n_k$
  - `(+^ n1...nk)`: All consequent formulas but $n_1, \ldots, n_k$

(Basic) Extended Expression Notation

- Term indexes:
  - `l,r`: Left- or right-hand side of a formula.
  - *n*: *n*-th term from left to right in a formula.
  - `-`*n*: *n*-th term from right to left in a formula.
  - `*`: All terms in a formula.
  - `(^` n_1 `...`n_k`)`: All terms in a formula but $n_1, \ldots, n_k$.

- Location references:
  - `(!` *fnum* `l|r` $i_1 \ldots i_n$`)`: Term in formula *fnum*, `l`eft- or `r`ight-hand side, at recursive path location $i_1 \ldots i_k$.

Examples

```
{-1} x * r + y * r + 1 >= r - 1
  |-------
{1} r = y * 2 * x + 1

>> (swap-rel -1)
{-1} r - 1 <= x * r + y * r + 1
  |-------
[1] r = y * 2 * x + 1

>> (swap!  (!  -1 r 1))
{-1} r - 1 <= r * x + y * r + 1
  |-------
[1] r = y * 2 * x + 1
```

Examples

```
{-1} x * r + y * r + 1 >= r - 1
  |-------
{1} r = y * 2 * x + 1

>> (swap-rel -1)
{-1} r - 1 <= x * r + y * r + 1
  |-------
[1] r = y * 2 * x + 1

>> (swap!  (!  -1 r 1))
{-1} r - 1 <= r * x + y * r + 1
  |-------
[1] r = y * 2 * x + 1
```

Examples

```
{-1} x * r + y * r + 1 >= r - 1
  |-------
{1} r = y * 2 * x + 1

>> (swap-rel -1)
{-1} r - 1 <= x * r + y * r + 1
  |-------
[1] r = y * 2 * x + 1

>> (swap!  (!  -1 r 1))
{-1} r - 1 <= r * x + y * r + 1
  |-------
[1] r = y * 2 * x + 1
```

Examples

```
{-1} r - 1 <= r * x + y * r + 1
  |-------
[1] r = y * 2 * x + 1

>> (group!  (!  1 r 1) r)

[-1] r - 1 <= r * x + y * r + 1
  |-------
{1} r = y * (2 * x) + 1

>> (flip-ineq -1)
  |-------
{1} r - 1 > r * x + y * r + 1
[2] r = y * (2 * x) + 1
```

Examples

```
{-1} r - 1 <= r * x + y * r + 1
  |-------
[1] r = y * 2 * x + 1

>> (group!  (!  1 r 1) r)

[-1] r - 1 <= r * x + y * r + 1
  |-------
{1} r = y * (2 * x) + 1

>> (flip-ineq -1)
  |-------
{1} r - 1 > r * x + y * r + 1
[2] r = y * (2 * x) + 1
```

Examples

```
{-1} r - 1 <= r * x + y * r + 1
  |-------
[1] r = y * 2 * x + 1

>> (group!  (!  1 r 1) r)

[-1] r - 1 <= r * x + y * r + 1
  |-------
{1} r = [y * (2 * x)] + 1

>> (flip-ineq -1)
  |-------
{1} r - 1 [>] r * x + y * r + 1
[2] r = y * (2 * x) + 1
```

```
{-1} r - 1 <= r * x + y * r + 1
  |-------
{1} r = y * (2 * x) + 1

>> (split-ineq -1)
{-1} r - 1 [=] r * x + y * r + 1
{-2} r - 1 <= r * x + y * r + 1
  |-------
{1} r = y * (2 * x) + 1

 >> (postpone)
{-1} r - 1 <= r * x + y * r + 1
  |-------
{1} r - 1 [=] r * x + y * r + 1
{2} r = y * (2 * x) + 1
```

```
{-1} r - 1 <= r * x + y * r + 1
  |-------
{1} r = y * (2 * x) + 1

>> (split-ineq -1)
{-1} r - 1 [=] r * x + y * r + 1
{-2} r - 1 <= r * x + y * r + 1
  |-------
{1} r = y * (2 * x) + 1

 >> (postpone)
{-1} r - 1 <= r * x + y * r + 1
  |-------
{1} r - 1 [=] r * x + y * r + 1
{2} r = y * (2 * x) + 1
```

```
{-1} r - 1 <= r * x + y * r + 1
  |-------
{1} r = y * (2 * x) + 1

>> (split-ineq -1)
{-1} r - 1 [=] r * x + y * r + 1
{-2} r - 1 <= r * x + y * r + 1
  |-------
{1} r = y * (2 * x) + 1

 >> (postpone)
{-1} r - 1 <= r * x + y * r + 1
  |-------
{1} r - 1 [=] r * x + y * r + 1
{2} r = y * (2 * x) + 1
```

More Strategies

Strategy	Description
`(mult-by` *fnums term*`)`	Multiply formula by term
`(div-by` *fnums term*`)`	Divide formula by term
`(move-terms` *fnum* `l|r` *tnums*`)`	Move additive terms left and right
`(isolate` *fnum* `l|r` *tnum*`)`	Isolate additive terms
`(cross-mult` *fnums*`)`	Perform cross-multiplications
`(factor` *fnums*`)`	Factorize formulas
`(factor!` *exprloc*`)`	Factorize terms
`(mult-eq` *fnum fnum*`)`	Multiply equalities
`(mult-ineq` *fnum fnum*`)`	Multiply inequalities

More Examples

```
{-1} (x * r + y) / pa > (r - 1) / pb
  |-------
{1} r - y * 2 * x = 1

>> (cross-mult -1)
{-1} pb * r * x + pb * y > pa * r - pa
  |-------
[1] r - y * 2 * x = 1

>> (isolate 1 l 1)
[-1] pb * r * x + pb * y > pa * r - pa
  |-------
{1} r = 1 + y * 2 * x
```

More Examples

```
{-1} (x * r + y) / pa > (r - 1) / pb
  |-------
{1} r - y * 2 * x = 1

>> (cross-mult -1)
{-1} pb * r * x + pb * y > pa * r - pa
  |-------
[1] r - y * 2 * x = 1

>> (isolate 1 1 1)
[-1] pb * r * x + pb * y > pa * r - pa
  |-------
{1} r = 1 + y * 2 * x
```

More Examples

```
{-1} (x * r + y) / pa > (r - 1) / pb
  |-------
{1} r - y * 2 * x = 1

>> (cross-mult -1)
{-1} pb * r * x + pb * y > pa * r - pa
  |-------
[1] r - y * 2 * x = 1

>> (isolate 1 l 1)
[-1] pb * r * x + pb * y > pa * r - pa
  |--------
{1} r = 1 + y * 2 * x
```

More Examples

```
{-1} x * y - pa + na < x * na * pa
{-2} r - y * 2 * x = 1
  |-------
{1} 2 * pa = 2 * x + 2 * y

>> (move-terms -1 1 (2 3))
{-1} x * y < x * na * pa + pa - na
[-2] r - y * 2 * x = 1
  |-------
[1] 2 * pa = 2 * x + 2 * y

>> (factor 1)
[-1] x * y < x * na * pa + pa - na
[-2] r - y * 2 * x = 1
  |-------
{1} 2 * pa = 2 * (x + y)
```

More Examples

```
{-1} x * y - pa + na < x * na * pa
{-2} r - y * 2 * x = 1
  |-------
{1} 2 * pa = 2 * x + 2 * y

>> (move-terms -1 1 (2 3))
{-1} x * y < x * na * pa + [pa] - [na]
[-2] r - y * 2 * x = 1
  |-------
[1] 2 * pa = 2 * x + 2 * y

>> (factor 1)
[-1] x * y < x * na * pa + pa - na
[-2] r - y * 2 * x = 1
  |-------
{1} [2 * pa = 2 * (x + y)]
```

More Examples

```
{-1} x * y - pa + na < x * na * pa
{-2} r - y * 2 * x = 1
  |-------
{1} 2 * pa = 2 * x + 2 * y

>> (move-terms -1 1 (2 3))
{-1} x * y < x * na * pa + [pa] - [na]
[-2] r - y * 2 * x = 1
  |-------
[1] 2 * pa = 2 * x + 2 * y

>> (factor 1)
[-1] x * y < x * na * pa + pa - na
[-2] r - y * 2 * x = 1
  |-------
{1} [2 * pa = 2 * (x + y)]
```

More Examples

```
[-1] x * y < x * na * pa + pa - na
[-2] r - y * 2 * x = 1
  |-------
{1} 2 * pa = 2 * (x + y)

>> (mult-eq -1 -2)
{-1} (x*y)*(r-y*2*x) < (x*n*pa+pa-na)*1
[-2] x * y < x * na * pa + pa - na
[-3] r - y * 2 * x = 1
  |-------
[1] 2 * pa = 2 * (x + y)

>> (mult-ineq -1 -2 (+ +))
{-1}
((x*y)*(r-y*2*x))*(x*y)<((x*na*pa+pa-na)*1)*(x*na*pa+pa-na)
...
  |-------
[1] 2 * pa = 2 * (x + y)
```

More Examples

```
[-1] x * y < x * na * pa + pa - na
[-2] r - y * 2 * x = 1
  |-------
{1} 2 * pa = 2 * (x + y)

>> (mult-eq -1 -2)
{-1} (x*y)*(r-y*2*x) < (x*n*pa+pa-na)*1
[-2] x * y < x * na * pa + pa - na
[-3] r - y * 2 * x = 1
  |-------
[1] 2 * pa = 2 * (x + y)

>> (mult-ineq -1 -2 (+ +))
{-1}
((x*y)*(r-y*2*x))*(x*y)<((x*na*pa+pa-na)*1)*(x*na*pa+pa-na)
...
  |-------
[1] 2 * pa = 2 * (x + y)
```

More Examples

```
[-1] x * y < x * na * pa + pa - na
[-2] r - y * 2 * x = 1
  |-------
{1} 2 * pa = 2 * (x + y)

>> (mult-eq -1 -2)
{-1} (x*y)*(r-y*2*x) < (x*n*pa+pa-na)*1
[-2] x * y < x * na * pa + pa - na
[-3] r - y * 2 * x = 1
  |-------
[1] 2 * pa = 2 * (x + y)

>> (mult-ineq -1 -2 (+ +))
{-1}
((x*y)*(r-y*2*x))*(x*y)<((x*na*pa+pa-na)*1)*(x*na*pa+pa-na)
...
  |-------
[1] 2 * pa = 2 * (x + y)
```

More Examples

```
...
  |-------
[1] 2 * pa = 2 * (x + y)

>> (div-by 1 "2")
...
  |-------
{1} pa = (x + y)

>> (mult-by 1 "100")
...
  |-------
{1} 100*pa = 100*(x + y)
```

More Examples

```
...
  |-------
[1] 2 * pa = 2 * (x + y)

>> (div-by 1 "2")
...
  |-------
{1} pa = (x + y)

>> (mult-by 1 "100")
...
  |-------
{1} 100*pa = 100*(x + y)
```

More Examples

```
...
  |-------
[1] 2 * pa = 2 * (x + y)

>> (div-by 1 "2")
...
  |-------
{1} pa = (x + y)

>> (mult-by 1 "100")
...
  |-------
{1} 100*pa = 100*(x + y)
```

Field

- Field is a PVS package for simplifications in the closed field of real numbers.
- `https://shemesh.larc.nasa.gov/fm/pvs/Field`.
- The package consists of:
  - The strategy `field`.
  - Several *extra-tegies*.

field

```
{-1} vox > 0
{-2} s * s - D*D > D
{-3} s * vix * voy - s * viy * vox /= 0
{-4} ((s * s - D*D) * voy - D * vox * sqrt(s*s - D*D))/
     (s * (vix * voy - vox * viy)) * s * vox /= 0
{-5} voy * sqrt(s * s - D*D) - D * vox /= 0
  |-------
{1} (viy * sqrt(s * s - D*D) - vix * D) /
    (voy * sqrt(s * s - D*D) - vox * D) =
    (D*D - s * s) / (((s * s - D*D) * voy - D * vox *
    sqrt(s * s - D*D)) /
    (s * (vix * voy - vox * viy)) * s * vox) +
    vix / vox

>> (field 1)
Q.E.D.
```

field

```
{-1} vox > 0
{-2} s * s - D*D > D
{-3} s * vix * voy - s * viy * vox /= 0
{-4} ((s * s - D*D) * voy - D * vox * sqrt(s*s - D*D))/
     (s * (vix * voy - vox * viy)) * s * vox /= 0
{-5} voy * sqrt(s * s - D*D) - D * vox /= 0
  |-------
{1} (viy * sqrt(s * s - D*D) - vix * D) /
    (voy * sqrt(s * s - D*D) - vox * D) =
    (D*D - s * s) / (((s * s - D*D) * voy - D * vox *
    sqrt(s * s - D*D)) /
    (s * (vix * voy - vox * viy)) * s * vox) +
    vix / vox

>> (field 1)
Q.E.D.
```

Some Extra-tegies

Strategy	Description
`(grind-reals)`	`grind` with `real_props`
`(cancel-by` *fnum term*`)`	Cancel a common term in a formula
`(skoletin` *fnum*`)`	Skolemize let-in expressions
`(skeep` *fnum*`)`	Skolemize with same variable names
`(neg-formula` *fnum*`)`	Negate a formula
`(add-formula` *fnum fnum*`)`	Add formulas
`(sub-formula` *fnum fnum*`)`	Subtract formulas

grind-reals

```
   |-------
{1} (x - 1 / 2) * (x - 1 / 2) >= 0

>> (grind-reals :dontdistrib 1)
Q.E.D.
```

grind-reals

```
   |-------
{1} (x - 1 / 2) * (x - 1 / 2) >= 0

>> (grind-reals :dontdistrib 1)
Q.E.D.
```

cancel-by

```
{-1} 4 * (pa * pb) + (pa * 6) * pa = pa * ((c + 1) * 2)
  |-------
{1} 2 * pb + 3 * pa = c

>> (cancel-by -1 "2*pa")
{-1} (3 * pa) + (2 * pb) = 1 + c
  |-------
{1} 2 * pa = 0
{2} 3 * pa + 2 * pb = c
```

cancel-by

```
{-1} 4 * (pa * pb) + (pa * 6) * pa = pa * ((c + 1) * 2)
  |-------
{1} 2 * pb + 3 * pa = c

>> (cancel-by -1 "2*pa")
{-1} (3 * pa) + (2 * pb) = 1 + c
  |-------
{1} 2 * pa = 0
{2} 3 * pa + 2 * pb = c
```

PVS's Let-in Expressions

- Let-in expressions are used in PVS to introduce local definitions.
- They are automatically unfolded by the theorem prover.

```
   |-------
{1} LET a = (x + 1), b = a * a, c = b * b IN c * c >= a

>> (assert)
   |-------
{1} 1 + x + (x*x*x*x*x*x*x*x + x*x*x*x*x*x*x*x)
    + (x*x*x*x*x*x*x + x*x*x*x*x*x*x)
    + (x*x*x*x*x*x*x + x*x*x*x*x*x*x)
    ...
    + (x*x + x)
    + (x*x + x)
    + (x*x + x)
    >= 1 + x
```

PVS's Let-in Expressions

- Let-in expressions are used in PVS to introduce local definitions.
- They are automatically unfolded by the theorem prover.

```
  |-------
{1} LET a = (x + 1), b = a * a, c = b * b IN c * c >= a

>> (assert)
  |-------
{1} 1 + x + (x*x*x*x*x*x*x*x + x*x*x*x*x*x*x*x)
    + (x*x*x*x*x*x*x + x*x*x*x*x*x*x)
    + (x*x*x*x*x*x*x + x*x*x*x*x*x*x)
    ...
    + (x*x + x)
    + (x*x + x)
    + (x*x + x)
    >= 1 + x
```

skoletin

```
   |-------
{1} LET a = (x + 1), b = a * a, c = b * b IN c * c >= a


>> (skoletin 1)
{-1} a = (x + 1)
   |-------
{1} LET b = a * a, c = b * b IN c * c >= a


>> (skoletin* 1)
{-1} c = b * b
{-2} b = a * a
[-3] a = (x + 1)
   |-------
{1} c * c >= a
```

skoletin

```
   |-------
{1} LET a = (x + 1), b = a * a, c = b * b IN c * c >= a

>> (skoletin 1)
{-1} a = (x + 1)
   |-------
{1} LET b = a * a, c = b * b IN c * c >= a

>> (skoletin* 1)
{-1} c = b * b
{-2} b = a * a
[-3] a = (x + 1)
   |-------
{1} c * c >= a
```

skoletin

```
  |-------
{1} LET a = (x + 1), b = a * a, c = b * b IN c * c >= a

>> (skoletin 1)
{-1} a = (x + 1)
  |-------
{1} LET b = a * a, c = b * b IN c * c >= a

>> (skoletin* 1)
{-1} c = b * b
{-2} b = a * a
[-3] a = (x + 1)
  |-------
{1} c * c >= a
```

More examples

```
  |-------
{1} FORALL (nnx:  nnreal, x:  real):
    nnx > x - nnx*nnx AND x + 2 * nnx*nnx >= 4 * nnx
    IMPLIES nnx > 1

>> (skeep)
{-1} nnx > x - nnx*nnx
{-2} x + 2 *nnx*nnx >= 4 * nnx
  |-------
{1} nnx > 1

>> (neg-formula -1)
{-1} nnx*nnx - x > -nnx
[-2] x + 2 * nnx*nnx >= 4 * nnx
  |-------
[1] nnx > 1
```

More examples

```
  |-------
{1} FORALL (nnx:  nnreal, x:  real):
    nnx > x - nnx*nnx AND x + 2 * nnx*nnx >= 4 * nnx
    IMPLIES nnx > 1

>> (skeep)
{-1} nnx > x - nnx*nnx
{-2} x + 2 *nnx*nnx >= 4 * nnx
  |-------
{1} nnx > 1

>> (neg-formula -1)
{-1} nnx*nnx - x > -nnx
[-2] x + 2 * nnx*nnx >= 4 * nnx
  |-------
[1] nnx > 1
```

More examples

```
  |-------
{1} FORALL (nnx:  nnreal, x:  real):
    nnx > x - nnx*nnx AND x + 2 * nnx*nnx >= 4 * nnx
    IMPLIES nnx > 1

>> (skeep)
{-1} [nnx] > [x] - [nnx]*[nnx]
{-2} [x] + 2 *[nnx]*[nnx] >= 4 * [nnx]
  |-------
{1} [nnx] > 1

>> (neg-formula -1)
{-1} [nnx*nnx - x > -nnx]
[-2] x + 2 * nnx*nnx >= 4 * nnx
  |-------
[1] nnx > 1
```

```
{-1} nnx*nnx - x > -nnx
[-2] x + 2 * nnx*nnx >= 4 * nnx
  |-------
[1] nnx > 1

>> (add-formulas -1 -2)
{-1} 3 * (nnx*nnx) > -nnx + 4 * nnx
  |-------
[1] nnx > 1

>> (cancel-by -1 "nnx")
Q.E.D.
```

```
{-1} nnx*nnx - x > -nnx
[-2] x + 2 * nnx*nnx >= 4 * nnx
  |-------
[1] nnx > 1

>> (add-formulas -1 -2)
{-1} 3 * (nnx*nnx) > -nnx + 4 * nnx
  |-------
[1] nnx > 1

>> (cancel-by -1 "nnx")
Q.E.D.
```

```
{-1} nnx*nnx - x > -nnx
[-2] x + 2 * nnx*nnx >= 4 * nnx
  |-------
[1] nnx > 1

>> (add-formulas -1 -2)
{-1} 3 * (nnx*nnx) > -nnx + 4 * nnx
  |-------
[1] nnx > 1

>> (cancel-by -1 "nnx")
Q.E.D.
```

Advanced Strategies

Importing	Scope
`Sturm@strategies`	Single-varible polynomial relations
`Tarski@strategies`	Boolean expressions of polynomial relations
`Hutch@strategies`	Boolean expressions of polynomial relations
`Bernstein@strategies`	Multi-variable polynomial relations
`affine_arith@strategies`	Multi-variable polynomial relations (rigorous approximations)
`interval_arith@strategies`	Real-valued functions (rigorous approximations)
`exact_real_arith@strategies`	Real-valued functions (arbitrary precision)
MetiTarski	Real-valued functions (external oracle)

sturm

Decision procedure based on Sturm's theorem

```
IMPORTING Sturm@strategies

sturm_fa :
  |-------
{1} FORALL(x:real):  x - x * x <= 1 / 4
>> (sturm)
Q.E.D.
sturm_ex :
  |-------
{1} EXISTS(x:real):  x >= 0 AND x^2 - x < 0
>> (sturm)
Q.E.D.
```

sturm

Decision procedure based on Sturm's theorem

```
IMPORTING Sturm@strategies

sturm_fa :
  |-------
{1} FORALL(x:real):  x - x * x <= 1 / 4
>> (sturm)
Q.E.D.
sturm_ex :
  |-------
{1} EXISTS(x:real):  x >= 0 AND x^2 - x < 0
>> (sturm)
Q.E.D.
```

sturm

Decision procedure based on Sturm's theorem

```
IMPORTING Sturm@strategies

sturm_fa :
  |-------
{1} FORALL(x:real):  x - x * x <= 1 / 4
>> (sturm)
Q.E.D.
sturm_ex :
  |-------
{1} EXISTS(x:real):  x >= 0 AND x^2 - x < 0
>> (sturm)
Q.E.D.
```

sturm

Decision procedure based on Sturm's theorem

```
IMPORTING Sturm@strategies

sturm_fa :
  |-------
{1} FORALL(x:real):  x - x * x <= 1 / 4
>> (sturm)
Q.E.D.
sturm_ex :
  |-------
{1} EXISTS(x:real):  x >= 0 AND x^2 - x < 0
>> (sturm)
Q.E.D.
```

tarski

Decision procedure based on Tarski's theorem

```
IMPORTING Tarski@strategies
tarski_fa :
  |-------
{1} FORALL (x:real):  (x-2)^2*(-x+4) > 0 AND
    x^2*(x-3)^2 >= 0 AND x-1 >= 0 AND -(x-3)^2+1 > 0
    IMPLIES -(x-11/12)^3*(x-41/10)^3 >= 0

>> (tarski)
Q.E.D.
tarski_ex :
  |-------
{1} EXISTS (x:real):  (x-2)^2*(-x+4) > 0 AND x^2*(x-3)^2 >= 0
    AND x-1 >= 0 AND -(x-3)^2+1 > 0
    AND -(x-11/12)^3*(x-41/10)^3 < 1/10

>> (tarski)
Q.E.D.
```

tarski

Decision procedure based on Tarski's theorem

```
IMPORTING Tarski@strategies
tarski_fa :
  |-------
{1} FORALL (x:real):  (x-2)^2*(-x+4) > 0 AND
    x^2*(x-3)^2 >= 0 AND x-1 >= 0 AND -(x-3)^2+1 > 0
    IMPLIES -(x-11/12)^3*(x-41/10)^3 >= 0

>> (tarski)
Q.E.D.
tarski_ex :
  |-------
{1} EXISTS (x:real):  (x-2)^2*(-x+4) > 0 AND x^2*(x-3)^2 >= 0
    AND x-1 >= 0 AND -(x-3)^2+1 > 0
    AND -(x-11/12)^3*(x-41/10)^3 < 1/10

>> (tarski)
Q.E.D.
```

tarski

Decision procedure based on Tarski's theorem

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    IMPLIES -(x-11/12)^3*(x-41/10)^3 >= 0

>> (tarski)
Q.E.D.
tarski_ex :
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{1} EXISTS (x:real):  (x-2)^2*(-x+4) > 0 AND x^2*(x-3)^2 >= 0
    AND x-1 >= 0 AND -(x-3)^2+1 > 0
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>> (tarski)
Q.E.D.
```

tarski

Decision procedure based on Tarski's theorem

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    x^2*(x-3)^2 >= 0 AND x-1 >= 0 AND -(x-3)^2+1 > 0
    IMPLIES -(x-11/12)^3*(x-41/10)^3 >= 0

>> (tarski)
Q.E.D.
tarski_ex :
  |-------
{1} EXISTS (x:real):  (x-2)^2*(-x+4) > 0 AND x^2*(x-3)^2 >= 0
    AND x-1 >= 0 AND -(x-3)^2+1 > 0
    AND -(x-11/12)^3*(x-41/10)^3 < 1/10

>> (tarski)
Q.E.D.
```

bernstein

Rigorous approximations using Bernstein polynomial basis

```
IMPORTING Bernstein@strategies
bernstein_fa :
  |-------
{1} FORALL (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 IMPLIES
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 +
    4*y^4 > -3.4

>> (bernstein)
Q.E.D.
bernstein_ex :
  |-------
{1} EXISTS (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 AND
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    < -3.39

>> (bernstein)
Q.E.D.
```

bernstein

Rigorous approximations using Bernstein polynomial basis

```
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bernstein_fa :
  |-------
{1} FORALL (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 IMPLIES
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 +
    4*y^4 > -3.4

>> (bernstein)
Q.E.D.
bernstein_ex :
  |-------
{1} EXISTS (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 AND
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    < -3.39

>> (bernstein)
Q.E.D.
```

bernstein

Rigorous approximations using Bernstein polynomial basis

```
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bernstein_fa :
  |-------
{1} FORALL (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 IMPLIES
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 +
    4*y^4 > -3.4

>> (bernstein)
Q.E.D.
bernstein_ex :
  |-------
{1} EXISTS (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 AND
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    < -3.39

>> (bernstein)
Q.E.D.
```

bernstein

Rigorous approximations using Bernstein polynomial basis

```
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bernstein_fa :
  |-------
{1} FORALL (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 IMPLIES
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 +
    4*y^4 > -3.4

>> (bernstein)
Q.E.D.
bernstein_ex :
  |-------
{1} EXISTS (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 AND
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    < -3.39

>> (bernstein)
Q.E.D.
```

affine

Rigorous approximations using affine arithmetic

```
IMPORTING affine_arith@strategies

affine_fa :

  |-------
{1} FORALL (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 IMPLIES
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    > -3.4

>> (affine)
Q.E.D.
affine_ex :

  |-------
{1} EXISTS (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 AND
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    < -3.39

>> (affine)
Q.E.D.
```

affine

Rigorous approximations using affine arithmetic

```
IMPORTING affine_arith@strategies

affine_fa :

  |-------
{1} FORALL (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 IMPLIES
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    > -3.4

>> (affine)
Q.E.D.
affine_ex :

  |-------
{1} EXISTS (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 AND
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    < -3.39

>> (affine)
Q.E.D.
```

affine

Rigorous approximations using affine arithmetic

```
IMPORTING affine_arith@strategies

affine_fa :

  |-------
{1} FORALL (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 IMPLIES
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    > -3.4

>> (affine)
Q.E.D.
affine_ex :

  |-------
{1} EXISTS (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 AND
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    < -3.39

>> (affine)
Q.E.D.
```

affine

Rigorous approximations using affine arithmetic

```
IMPORTING affine_arith@strategies

affine_fa :

  |-------
{1} FORALL (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 IMPLIES
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    > -3.4

>> (affine)
Q.E.D.
affine_ex :

  |-------
{1} EXISTS (x,y:real):
    -0.5 <= x AND x <= 1 AND -2 <= y AND y <= 1 AND
    4*x^2 - (21/10)*x^4 + (1/3)*x^6 + (x-3)*y - 4*y^2 + 4*y^4
    < -3.39

>> (affine)
Q.E.D.
```

aa-numerical

Numerical approximations using affine arithmetic

```
{-1} -0.5 <= x
{-2} x <= 1
{-3} -2 <= y
{-4} y <= 1
  |-------
{1} 4*x^2-(21/10)*x^4+(1/3)*x^6+(x-3)*y-4*y^2+4*y^4 > -3.4

>> (aa-numerical (!  1 1) :precision 5)

{-1} 4*x^2-(21/10)*x^4+(1/3)*x^6+(x-3)*y-4*y^2 + 4*y^4
    ## [|-3.43158, 55.90987|]
...
  |-------
...
```

aa-numerical

Numerical approximations using affine arithmetic

```
{-1} -0.5 <= x
{-2} x <= 1
{-3} -2 <= y
{-4} y <= 1
  |-------
{1} 4*x^2-(21/10)*x^4+(1/3)*x^6+(x-3)*y-4*y^2+4*y^4 > -3.4

>> (aa-numerical (!  1 l) :precision 5)

{-1} 4*x^2-(21/10)*x^4+(1/3)*x^6+(x-3)*y-4*y^2 + 4*y^4
    ## [|-3.43158, 55.90987|]
...
  |-------
...
```

interval

Rigorous approximations using interval arithmetic

```
IMPORTING interval_arith@strategies

sin_x_cos :

  |-------
{1} EXISTS (d:  real):
    d ## [|0, 90|] AND sin(d*pi/180)*cos(d*pi/180) <= 1/2

>> (interval)
Q.E.D.
tr_200_250_abs_35 :

{-1} abs(phi) <= 35
{-2} v ## [|200, 250|]
  |-------
{1} abs(((180*g)/(pi*v*0.514))*tan((pi*phi)/180)) <= 3.825

>> (interval)
Q.E.D.
```

interval

Rigorous approximations using interval arithmetic

```
IMPORTING interval_arith@strategies

sin_x_cos :

  |-------
{1} EXISTS (d:  real):
    d ## [|0, 90|] AND sin(d*pi/180)*cos(d*pi/180) <= 1/2

>> (interval)
Q.E.D.
tr_200_250_abs_35 :

{-1} abs(phi) <= 35
{-2} v ## [|200, 250|]
  |-------
{1} abs(((180*g)/(pi*v*0.514))*tan((pi*phi)/180)) <= 3.825

>> (interval)
Q.E.D.
```

interval

Rigorous approximations using interval arithmetic

```
IMPORTING interval_arith@strategies

sin_x_cos :

  |-------
{1} EXISTS (d:  real):
    d ## [|0, 90|] AND sin(d*pi/180)*cos(d*pi/180) <= 1/2

>> (interval)
Q.E.D.
tr_200_250_abs_35 :

{-1} abs(phi) <= 35
{-2} v ## [|200, 250|]
  |-------
{1} abs(((180*g)/(pi*v*0.514))*tan((pi*phi)/180)) <= 3.825

>> (interval)
Q.E.D.
```

interval

Rigorous approximations using interval arithmetic

```
IMPORTING interval_arith@strategies

sin_x_cos :

  |-------
{1} EXISTS (d:  real):
    d ## [|0, 90|] AND sin(d*pi/180)*cos(d*pi/180) <= 1/2

>> (interval)
Q.E.D.
tr_200_250_abs_35 :

{-1} abs(phi) <= 35
{-2} v ## [|200, 250|]
  |-------
{1} abs(((180*g)/(pi*v*0.514))*tan((pi*phi)/180)) <= 3.825

>> (interval)
Q.E.D.
```

numerical

Numerical approximations using interval arithmetic

```
{-1} abs(phi) <= 35
{-2} v ## [|200, 250|]
  |-------
{1} abs(((180*g)/(pi*v*0.514))*tan((pi*phi)/180)) <= 3.825

>> (numerical (!  1 1) :precision 5)

{-1} abs(((180*g)/(pi*v*0.514))*tan((pi*phi)/180)) ##
      [|0, 3.82457|]
...
  |-------
...
```

numerical

Numerical approximations using interval arithmetic

```
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...
  |-------
...
```

era-numerical

Exact real arithmetic

```
IMPORTING exact_real_arith@strategies

sqrt_pi :

  |-------
{1} sqrt(pi) < 2

>> (era-numerical (!  1 1) :precision 20)

{-1} sqrt(pi) < 1.77245385090551602731
{-2} 1.77245385090551602729 < sqrt(pi)
  |-------
{1} sqrt(pi) < 2
```

era-numerical

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{1} sqrt(pi) < 2
```

metit

Using MetiTarski as an external oracle

```
Ayad_Marche :


  |-------
{1} FORALL (r:real):  abs(r) <= 1 IMPLIES
    abs(0.9890365552+1.130258690*r+0.5540440796*r*r-exp(r))
    <= (1-2^-16)*2^-4


>> (metit)
MetiTarski Input = fof(pvs2metit,conjecture, (![R1]:
((abs(R1) <= 1)
=> (abs(((((9890365552 / 10000000000) + ((1130258690 /
1000000000) *
R1)) + (((5540440796 / 10000000000) * R1) * R1)) - exp(R1)))
<=
((1 - 2^-16) * 2^-4))))).
SZS status Theorem for Ayad_Marche.tptp
Processor time:  0.081 = 0.048 (Metis) + 0.033 (RCF)
Maximum weight in proof search:  424
MetiTarski succesfully proved.
Trusted oracle:  MetiTarski.
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```

mono-poly

Dicharges monocity properties of polynomials

```
mono_fa :
  |-------
{1} FORALL (x,y:real):
    x >= 1 AND x < y IMPLIES
    (x - 1/4) ^ 2 <= y*y - (1/2 *y + (1/16)

>> (mono-poly)
Q.E.D.
mono_ex :
  |-------
{1} EXISTS (x,y:real):  x < y AND x^2 >= sq(y)

>> (mono-poly)
Q.E.D.
```

mono-poly

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